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Math 2050 week 1:

<u>Goal</u>: Study $\mathbb{R}$, the set of real no.

<u>Question</u>: what is $\mathbb{R}$??

<u>Warm-up</u>: What is $\mathbb{N}$?? natural number.

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

Algebra on $\mathbb{N}$: addition $+$, multiplication $\cdot$

$(\mathbb{N}, +, \cdot)$

How to characterise $\mathbb{N}$ as a set of algebra "addition"??

① non-emptyness: well-ordering property

$\forall\, S \subseteq \mathbb{N}$, $\exists\, m \in S$ s.t. $m \leq s$ $\forall s \in S$ — what is it??

② Induction property: $\mathbb{N}$ is s.t. if $m \in \mathbb{N}$, then $m+1 \in \mathbb{N}$.

Principle of MI: If $S \subseteq \mathbb{N}$ is s.t. $1 \in S$ and ② holds then $S = \mathbb{N}$.

pf: If not, $S \subsetneq \mathbb{N}$ $\Rightarrow$ $\mathbb{N} \setminus S \neq \phi$.

$\Rightarrow \exists$ least element $l \in \mathbb{N} \setminus S$ i.e. $l \notin S$.

$\Rightarrow$ $l-1 \in S$ (since $l-1 \notin \mathbb{N} \setminus S$)

$\overset{②}{\Longrightarrow}$ $l = l-1+1 \in S \cap S^c$ $\rightarrow\leftarrow$.

Conseq: "Pf by MI" $\Leftrightarrow$ proving the statement $\forall n$

$\Leftrightarrow$ $\{n \mid \text{statement holds}\} = \mathbb{N}$.

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Subset of $\mathbb{N}$:

eg: 1. $\{1, 4, 7\}$ $\to$ finite set

2. $\{2, 4, 6, 8, \ldots\} = 2\mathbb{N}$ $\to$ infinite set (countable)

$\{1, 2, 3, 4, \ldots\}$ $\quad f : \mathbb{N} \to 2\mathbb{N}$ given by $x \mapsto 2x$

is a bijection (pairing up)

Def: A set S is countably ∞ if $\exists f : \mathbb{N} \to S$ which is bijection.

"Generalization": Consider $(\mathbb{N}, +)$, bad as a algebra (why?)

eg: $1 + 2 = 3$ (reverse) ??

negation of $(\mathbb{N})$

Want: $3 + (-2) = 1$

But! $-2 \notin \mathbb{N}$.

$\rightsquigarrow$ def $\mathbb{Z} = \left\{ \begin{matrix} 0, 1, 2, \ldots \\ -1, -2, \ldots \end{matrix} \right\}$ s.t. $(\mathbb{Z}, +) =$ group.

i.e. $\mathbb{Z}$ obeys: (a1): $a + b = b + a \quad \forall a, b \in \mathbb{Z}$

(a2): $(a+b) + c = a + (b+c) \quad \forall a, b, c \in \mathbb{Z}$

(a3): $\exists\ 0 \in \mathbb{Z}$ s.t. $a + 0 = a \quad \forall a \in \mathbb{Z}$.

(a4): $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$ s.t.

$a + b = 0$.

(call b to be $-a$).

Now if we declare $\mathbb{N}$ to be set of the number

then we can define $a > b$ for $a, b \in \mathbb{Z}$

if $a + (-b) \in \mathbb{N}$.

(≥ 0 if $a + (-b) \in \mathbb{N} \cup \{0\}$)

Check!

Multiplication on $\mathbb{N}$: $a \cdot b \triangleq \underbrace{a + a + \cdots + a}_{b}$

on $\mathbb{Z}$: $a \cdot b \triangleq \begin{cases} \underbrace{a + \cdots + a}_{b} & \text{if } b > 0 \\ -\underbrace{(a + \cdots + a)}_{-b} & \text{if } b < 0 \end{cases}$

Analogy of (a1) – (a4) : fail !!

$3 \cdot 4 = 12 \longrightarrow 3 = 12 \cdot 4^{-1}$ $\notin \mathbb{Z}$.

Introduce $\mathbb{Q}$: the set of rational number.

$\mathbb{Q}$ " = set where (M1 – M4) holds "

$= \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \}$

($\frac{m}{n}$ is defined to be s.t. $\frac{m}{n} \cdot n = m$.)

M1: $\forall a, b \in \mathbb{Q}$, $a \cdot b = b \cdot a$

M2: $\forall a, b, c \in \mathbb{Q}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M3: $\exists 1 \in \mathbb{Q}$ s.t. $a \cdot 1 = a$ $\forall a \in \mathbb{Q}$

M4: $\forall a \in \mathbb{Q} \setminus \{0\}$, $\exists b \in \mathbb{Q}$ s.t. $a \cdot b = b \cdot a = 1$.

(d) : $\forall a, b, c \in \mathbb{Q}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

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In term of Axioms :

$\boxed{\text{Going back to } \mathbb{R}}$ (must be a set of element sharing the same set of algebraic properties as $\mathbb{Q}$)

$\mathbb{R}$: satisfies (a1)-(a4) and (M1)-(M4) and (d).

(Before we move on)

Thm : Cors a conseq of axioms)

(1) if $a, b \in \mathbb{R}$ s.t. $a+b = a$, then $b = 0$

(2) if $a, b \in \mathbb{R}$ s.t. $a \cdot b = a \neq 0$ then $b = 1$.

(3) Given $a \in \mathbb{R}$, if $b, c \in \mathbb{R}$ s.t. $a+b = a+c$, then $b = c$

(4) Given $a \neq 0 \in \mathbb{R}$, if $b, c \in \mathbb{R}$ s.t. $a \cdot b = a \cdot c$ then $b = c$

pf: (1) : $a+b = a$ ($\forall a, \exists a' \in \mathbb{R}$ s.t. $a + a' = 0$)

$\Rightarrow a' + (a+b) = a' + a = 0$

$\Rightarrow (a'+a) + b = 0 + b = b$. #.

Hence 0, 1 are unique "Identity".

(3) : $a+b = a+c \Rightarrow a' + (a+b) = (a'+a) + b = b$

$a' + (a+c) = c$

hence $a' + a = 0 = a'' + a \Rightarrow a' = a''$

ie. the inverse of a is unique

Hence it is canonical to call it $\underline{\underline{-a}}$ / as well as a^{-1}

As in $\mathbb{Z}$ (or $\mathbb{Q}$),
~~To define a~~ To define a ordering $a \geq b$, suffices to define the set of positive numbers.

Given $\mathbb{R}$ (although unclear)

let $\mathbb{P} \subseteq \mathbb{R}$ be subset s.t.

① if $a \in \mathbb{R}$, then either $a = 0$, $a \in \mathbb{P}$ or $-a \in \mathbb{P}$.

② if $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$

③ if $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.

Also define $|a|$ for $a \in \mathbb{R}$ by

$$|a| = \begin{cases} a & \text{if } a \in \mathbb{P} \cup \{0\} \\ -a & \text{if } -a \in \mathbb{P} \end{cases}$$

Distinction of $\mathbb{R}$ and $\mathbb{Q}$??

- So far $\mathbb{Q}$ satisfies all required properties (all algebraic)
- Problem (or question): Can we solve non-linear eqn in $\mathbb{Q}$??

<u>eg</u>: Can we find $x \in \mathbb{Q}$ s.t. $x^2 = 2$??

<u>Thm</u>: $\nexists\, x \in \mathbb{Q}$ s.t. $x^2 = 2$.

pf: Suppose $x = \frac{m}{n}$ s.t. $\left(\frac{m}{n}\right)^2 = 2$, $(m, n) = 1$ coprime.

$$m^2 = 2n^2 \Rightarrow \begin{cases} m = \text{even} \\ n = \text{even} \end{cases} \rightarrow\leftarrow$$

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Solution to Problem: extending $\mathbb{Q}$ to $\mathbb{R}$.

But how ??

eg: how to make sense of $\sqrt{2}$?? (as operation from $\mathbb{Q}$??)

* Cannot define one-by-one !! (Not everything from algebraic equation)

* $\sqrt{2} \approx \underbrace{1.41421356237\ldots\ldots}$ concept of limit

something s.t. $m^2 \boxed{\approx} 2$.

Completeness of $\mathbb{R}$ (extra properties from $\mathbb{Q}$)

Defn: Given $S \subseteq \mathbb{R}$, S is bdd from above if $\exists M \in \mathbb{R}$ s.t. $s \leq M \quad \forall s \in S$.

Defn (the best upper bdd) Given $S \subseteq \mathbb{R}$ which is bdd from above, u is said to be ~~sup(S)~~ least upper bdd if.

① $u \geq s \quad \forall s \in S$

② if $\exists v \in \mathbb{R}$ s.t. $s \leq v \quad \forall s \in S$ then $u \leq v$.

Conseq: if $u, \tilde{u}$ are least upper bdd

$\Rightarrow u \leq \tilde{u}$ and $\tilde{u} \leq u$

$\Rightarrow u = \tilde{u} \rightarrow$ canonical to call it $\sup(S)$.

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Rmk: might define $\inf S$ as

① $u \le s \quad \forall s \in S$

② if $v \le s \quad \forall s \in S$, then $v \le u$ #

Completeness axiom of $\mathbb{R}$: (complete the "gap").

if $S \subseteq \mathbb{R}$ s.t. S is bdd from above,

then $\exists u \in \mathbb{R}$ s.t. $u = \sup(S)$.

$\Updownarrow$

if $S \subseteq \mathbb{R}$ s.t. S is bdd from below

then $\exists u \in \mathbb{R}$ s.t. $\inf S = u$.

Thm: $\mathbb{N}$ is unbdd. (Archimedean property)

pf: if Not, then $\exists \sup \mathbb{N} = \bar{n} \in \mathbb{R}$.

$\sup \mathbb{N} = \bar{n} \Rightarrow \bar{n}+1 \notin \mathbb{N}$. $\rightarrow\leftarrow$ with Induction #

Thm: $\exists u \in \mathbb{R}$ s.t. $u^2 = 2$

pf: Consider $S = \{ x \in \mathbb{R} \mid x^2 < 2 \}$

① $S \neq \phi$ since $0^2 = 0 < 2$

② if $x \in S$, then $x \le 2$ since otherwise

$x_0 > 2$ for some $x_0 \in S$

$\Rightarrow 2 > x_0^2 > 4 \quad \rightarrow\leftarrow$.

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Completeness $\Rightarrow$ $\exists\, u = \sup(S)$, $u \geq \phi$ since $\phi \in S$.

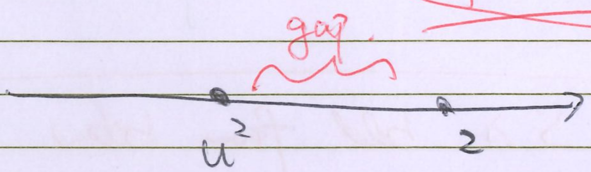
$\vdash$: $u^2 = 2$.

① Suppose $u^2 < 2$,

(Want to draw contradiction! $u \neq$ upper bdd)

Let $\varepsilon > 0$ $\quad (u+\varepsilon)^2 = u^2 + 2u\varepsilon + \varepsilon^2$

fill the gap



Choose $\varepsilon > 0$ sufficiently small s.t.

$u^2 + 2u\varepsilon + \varepsilon^2 < 2$.

this can be done if $\varepsilon(2u+\varepsilon) < 2 - u^2$.

$$\Uparrow$$

$$\varepsilon(2u+\varepsilon) \leq \varepsilon(2u+1) < 2-u^2$$

if $\varepsilon < 1$

so we choose $\varepsilon = \min\left\{1, \frac{2-u^2}{2u+1}\right\} > 0$.

$\Rightarrow$ $(u+\varepsilon)^2 < 2$ for some small $\varepsilon > 0$.

$\Rightarrow$ $u+\varepsilon \in S$ and $u+\varepsilon \leq u$

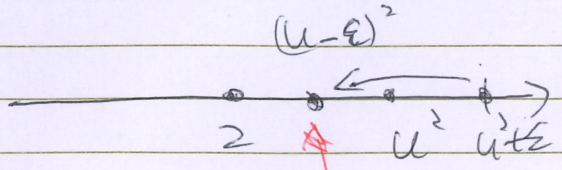
$\Rightarrow\Leftarrow$.

※ Not allow to use $\sqrt{\ }$ operation.

② Suppose $u^2 > 2$ (want draw contradiction from u = least upper bound)

Let $0 < \varepsilon \ll 1$

$$(u-\varepsilon)^2 = \overset{>2}{u^2} - 2\varepsilon u + \varepsilon^2 .$$



if $\varepsilon \ll 1$

$(u-\varepsilon)^2 > 2$

$\Rightarrow$ $u-\varepsilon$ is an upper bound of S.

since otherwise $\exists\, x \in S$ s.t.

$x > u-\varepsilon > 0$

$\Rightarrow 2 > x^2 > (u-\varepsilon)^2 > 2 \Rightarrow\leftarrow$

$\Rightarrow u-\varepsilon \geq u \Rightarrow\leftarrow$

Identify ε: require $u^2 - 2\varepsilon u \cancel{+\varepsilon^2}$

$> u^2 - 2\varepsilon u > 2$

Choosing $0 < \varepsilon = \dfrac{u^2-2}{2u}$. #.

Alternative formation of $\sup(S) = u$

① $u \geq s \quad \forall s \in S$

② $\forall \varepsilon > 0$, $\exists\, s_\varepsilon \in S$ s.t.

$u - \varepsilon < s_\varepsilon$

(since $u-\varepsilon \neq$ upper bound $\Rightarrow \exists\, s_\varepsilon > u-\varepsilon$)